NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1394

A FLAT WING WITH SHARP EDGES IN A SUPERSONIC STREAM

By A. E. Donov

Translation of Izvestiia-Akademia, NAUK, USSR, 1939.

Washington
March 1956

1J

NATIONAL ADVISORY COMMITTEE FOR AERONAUTI...

TECHNICAL MEMORANDUM 1394

A FLAT WING WITH SHARP EDGES IN A SUPERSONIC STREAM*

By A. E. Donov

In this work there is given an approximate solution of the problem of a two-dimensional steady supersonic stream of ideal gas, neglecting heat conduction, flowing around a thin wing with sharp edges at small angles of attack. (Determination of the law of distribution of pressure along the wing, lifting force and head resistance of the wing.)

PART I

The problem of the investigation of the mechanical action of a moving gas on an immovable wing appears as a special case of the somewhat more general problem of the investigation of the mechanical action of a moving gas on an immovable fixed wall constraining the motion of the gas. In our own explanation we begin with the formulation of this last problem in which we confine ourselves only to the consideration of the steady two-dimensional forces of ideal gases not subject to the action of gravitational forces. In the plane of motion of the gas we shall arrange an immovable rectangular coordinate system in such a manner that it is situated as in figure 1. We introduce three functions $\vec{v}$, ρ, and p of the independent variables x and y defined, respectively, as the velocity, density, and pressure. The vector functions $\vec{v}$ will be determined by a pair of scalar functions of the independent variables. For these functions we shall agree to take either the functions v_x, v_y defined as the projections of the velocity of the axis x and y, respectively, or the functions v and β, defined, respectively, as the absolute value of the velocity and its angle with respect to the positive direction of the x-axis, measured in the counterclockwise sense. In what follows we limit ourselves to the consideration only of flows for which the function β satisfies the condition

$$-\frac{\pi}{2} < \beta < \frac{\pi}{2} \tag{1}$$

*Izvestiia-Akademia, NAUK, USSR, 1939, pp 603-626.

As is well known, the study of the gas motion under consideration leads to the investigation of the following system of differential equations

$$\left.\begin{aligned} & v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \\ & v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \\ & \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} = 0 \\ & v_x \frac{\partial}{\partial x}\left(\frac{p}{\rho^k}\right) + v_y \frac{\partial}{\partial y}\left(\frac{p}{\rho^k}\right) = 0 \end{aligned}\right\} \qquad (2)$$

Here k is the adiabatic exponent (for air $k = 1.405$). If the motion of the gas is constrained by an immovable frictionless fixed wall in the plane XOY, the gas will be adjacent to it along some curve. We shall call this curve the "contour K."

Consider the unit vector $\vec{t}$ tangent to the contour K directed in such a manner that its projection on the x-axis is positive. Denote by β_k the angle which it makes with the x-axis. Clearly β_k may be regarded as a function of the abscissa x of that point of the contour K associated with the vector $\vec{t}$. We denote this function by $\beta_k(x)$ and assume that it is continuous. If the function $\beta_k(x)$ is prescribed and, moreover, the coordinates of any point of the contour K are given, the form and position of the contour is completely determined. We agree to take as origin the left edge of the contour K. Then the equation of this contour will have the form

$$y = \int_0^x \tan \beta_k(x)\,dx \qquad (3)$$

We can write this equation more briefly if we designate its right-hand side by $y_k(x)$

$$y = y_k(x) \tag{4}$$

Since in the flows under consideration the direction of the velocity on the contour K must coincide with the vector $\vec{t}$, the condition on the flow along an immovable fixed frictionless wall may be written in the following fashion

$$\beta = \beta_k(x) \tag{5}$$

at $y = y_k(x)$. The condition (5) must be added to the system of equations (2) as a qualifying boundary condition. Much work has been dedicated to the investigation of solutions of the system (2) subject to the condition (5). Of these we are interested here only in those in which the flow is supersonic, i.e., flows at every point of which the following condition

$$v > a \tag{6}$$

is satisfied, where a is the local speed of sound

$$a = \sqrt{k \frac{p}{\rho}} \tag{7}$$

The investigations contained in these works divide in two fundamental directions. The first direction is represented in works in which solutions of the problem are achieved with the help of numerical or graphical processes permitting the step-by-step calculation of a system of particular values of the desired functions. (Works of Busemann, Kibelia, and Frankl.) The fundamental achievements of the methods represented by these works consist of the fact that by their use many actual practical problems may be solved quantitatively of which the solution by other methods would present great difficulties. In particular these methods solve thoroughly corner-nonpotential problems. The chief defect of these methods is that the solutions obtained are numerical so that it is impossible to obtain a general qualitative estimate of the phenomena under investigation. The second direction is represented by the works of Meyer, Ackeret, Prandtl, and Busemann, which are confined to a cultivation of an exact theory of irrotational flows. The results are based on the fact that in the case where vorticity is absent the characteristic system of differential equations (2) admit of integrable combinations. This theory leads to series of approximate results of any desired accuracy, giving a complete qualitative and quantitative picture of the flow. Since our investigation is mostly connected with the theory of irrotational flows we give below a brief introduction to the fundamental methods and results of this theory.

We introduce the stream function ψ defined by the following relations

$$\left.\begin{aligned} \frac{\partial\psi}{\partial x} &= \rho v_y \\ \frac{\partial\psi}{\partial y} &= -\rho v_x \end{aligned}\right\} \tag{8}$$

As is well known from equations (2), (7), and (8) the following relations follow without difficulty

$$\frac{p}{\rho^k} = \theta \tag{9}$$

$$\frac{v^2}{2} + \frac{a^2}{k - 1} = t_0 \tag{10}$$

Where θ, t_0 denote quantities which display the flow once and for all as a function only of ψ. With the help of equations (2), (9), and (10) it is easy to obtain the two equations

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \Omega \tag{11}$$

$$(a^2 - v_x^2)\frac{\partial v_x}{\partial x} + (a^2 - v_y^2)\frac{\partial v_y}{\partial y} - v_x v_y\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) = 0 \qquad (12)$$

where Ω denotes a quantity defined as

$$\Omega = \rho\left[\frac{dt_0}{d\psi} - \frac{a^2}{k(k - 1)}\frac{d \ln \theta}{d\psi}\right] \qquad (13)$$

Equations (11) and (12) represent linear relationships between the first partial derivatives of the functions v_x, v_y with respect to x and y.

Since every flow under consideration is supersonic, the entire region of the flow may be covered by a pair of families of characteristics. The differential equations of these characteristics are obtained easily by the use of equations (11) and (12). For one family of characteristics, which we shall agree to call the first family, we obtain the equations

$$dy = m_1 dx \qquad (14)$$

$$d(v \cos \beta) + m_2 d(v \sin \beta) = \Omega \frac{(a^2 - v^2\cos^2\beta)m_1 + v^2\sin \beta \cos \beta}{v^2\cos^2\beta - a^2} dx \qquad (15)$$

and for the other, which we shall agree to call the second family, we have the equations

$$dy = m_2 dx \qquad (16)$$

$$d(v \cos \beta) + m_1 d(v \sin \beta) = \Omega \frac{(a^2 - v^2\cos^2\beta)m_2 + v^2\sin \beta \cos \beta}{v^2\cos^2\beta - a^2} dx \qquad (17)$$

Here m_1, m_2 denote the following expressions

$$m_1 = \frac{-v^2\sin \beta \cos \beta + a\sqrt{v^2 - a^2}}{a^2 - v^2\cos^2\beta} \qquad (18)$$

$$m_2 = \frac{-v^2 \sin\beta\cos\beta - a\sqrt{v^2 - a^2}}{a^2 - v^2\cos^2\beta} \tag{19}$$

We now consider a supersonic stream with constant hydrodynamical elements (i.e., functions v, β, ρ, p, a). We shall call this flow the undisturbed flow. The values of the functions v, ρ, p, a in the undisturbed stream will be denoted by w, ρ_0, p_0, a_0 respectively, and the ratio w/a_0 by M. Since the stream under consideration is supersonic, $M > 1$. We shall choose the direction of the velocity of the undisturbed stream to correspond to the direction of the x-axis.

We assume that the undisturbed stream strikes an immovable, fixed, frictionless wall (contour K), inclined in such a manner that in flowing around this wall the stream never detaches from it and remains supersonic everywhere. We may distinguish two cases of flows of this type.

Case I.- The contour K is situated in such a manner that the condition

$$\beta_k(0) \leq 0 \tag{20}$$

is fulfilled. In this case, as is well known, there appears a curve of weak discontinuity OC (figs. 2(a), and 2(b)) proceeding from the origin and dividing the entire flow in two parts. On one side of the curve of weak discontinuity OC extends the region containing the undisturbed stream and on the other the region of flow around the wall. In the region of flow around the fixed frictionless wall the hydrodynamical elements of the stream, generally speaking, are not constant but vary. In what follows we shall call this part of the stream the disturbed stream. In the entire region of the flow under consideration the functions v, β, ρ, p, a are continuous but their partial derivatives with respect to x and y (all or only some) exhibit jump discontinuities, at least on the curve of weak discontinuity OC. The same curve OC appears as a characteristic of the second family since the hydrodynamical elements of the stream are constant. On this line the following relationships will hold in the entire region containing the stream

$$t_0 = \frac{w^2}{2} + \frac{a_0{}^2}{k - 1} \tag{21}$$

$$\theta = \theta_0 \tag{22}$$

where θ_0 denotes a quantity defined as

$$\theta_0 = \frac{p_0}{{\rho_0}^k} \tag{23}$$

From equations (13), (21), and (22) we easily obtain

$$\Omega = 0 \tag{24}$$

i.e., the flow under consideration is irrotational. By virtue of relation (24) the right-hand side of equations (15) and (17) vanish and these equations can be integrated. As a result of integration of equation (15) we obtain the relationship

$$\beta + \varphi(v) = \text{constant} \tag{25}$$

satisfied along any characteristic of the first family, and as a result of integrating equation (17) we have

$$\beta - \varphi(v) = \text{constant} \tag{26}$$

satisfied along a characteristic of the second family. $\varphi(v)$ denotes a function defined as

$$\varphi(v) = \sqrt{\frac{k + 1}{k - 1}} \text{ arc tan} \sqrt{\frac{k - 1}{k + 1}} \frac{\sqrt{v^2 - a^2}}{a} - \text{arc tan} \frac{\sqrt{v^2 - a^2}}{a} \tag{27}$$

Since on the curve of weak discontinuity OC the quantities v and β have the values w and 0, respectively, the following relation is satisfied along every characteristic of the first family intersecting this line and consequently in the entire region of the disturbed stream:

$$\beta + \varphi(v) = \varphi(w) \tag{28}$$

From equations 16, 26, and 28 it immediately follows that the characteristics of the second family (the curve OC being among these) are straight lines since along each of these characteristics the hydrodynamical elements are constant.

Making use of these circumstances it is not difficult with the aid of equations 28, 26, 22, 21, 16, 10, 7, and 5 to construct expressions for the functions v, β, ρ, p, a in the region of the disturbed flow. However, the construction of these expressions is not of great interest since our chief interest is centered on the construction of an expression for the pressure on the contour K which may be accomplished without the use of these expressions for the hydrodynamical elements of the flow. Actually equation 28 allows us to determine the velocity v as a function of the angle of inclination of this velocity with the x-axis at every point of the region filled by the disturbed flow. Since by virtue of equation 5 the angle of inclination of the velocity with respect to the x-axis is a given function of x on the contour K there is the possibility of using equations 22, 21, 10, 9, and 7 to determine the pressure p as a function of x on the flow around a contour. If we limit ourselves to the consideration of slightly disturbed flows, i.e., flows whose hydrodynamical elements differ but little from the hydrodynamical elements of the undisturbed flow, the expression for the pressure on the flow around a contour K may be written in the form of a series. This series has the form

$$p = p_0 + q\left[a_1\beta_k(x) + a_2\beta_k^{\,2}(x) + a_3\beta_k^{\,3}(x) + a_4\beta_k^{\,4}(x) + \ldots\right] \tag{29}$$

where

$$q = \frac{\rho_0 w^2}{2} = \frac{p_0 k M^2}{2}$$

$$a_1 = 2(M^2 - 1)^{-1/2}$$

$$a_2 = (M^2 - 1)^{-2}\left(2 - 2M^2 + \frac{k + 1}{2} M^4\right)$$

$$a_3 = (M^2 - 1)^{-7/2}\left[\frac{4}{3} - 2M^2 + \frac{5}{3}(k + 1)M^4 + \frac{-5 - 7k + 2k^2}{6} M^6 + \frac{k + 1}{6} M^8\right]$$

$$a_4 = (M^2 - 1)^{-5}\left(\frac{1}{3} - \frac{2}{3} M^2 + \frac{7 + 19k}{6} M^4 + \frac{-21 - 43k + 18k^2}{12} M^6 + \frac{15 + 20k - 8k^2 + 3k^3}{12} M^8 + \frac{-21 - 20k + 3k^2 + 2k^3}{48} M^{10} + \frac{3 + 2k - k^2}{48} M^{12}\right)$$

. . . .

Case II.- The flow around a contour K is situated in such a manner that the following inequality is satisfied

$$\beta_k(0) > 0 \tag{30}$$

In this case, as is well known, a line of shock discontinuity OD appears (fig. 3) proceeding from the origin O and dividing the entire flow under consideration in two parts. On one side of this line is the region of the undisturbed stream and on the other the region in which the fluid flows around the fixed frictionless wall. Just as in case I we call the flow in the region in which the stream around the fixed frictionless wall is accomplished the disturbed flow. In the present case, in contrast to case I, the functions v, β, ρ, p, a exhibit jump discontinuities on the shock-line OD.

In the region of the disturbed flow these functions must satisfy not only equations 2, 5, and 7 but also the dynamical conditions across the shock line. Considering the flow to be only slightly disturbed, these conditions may be written in the following form

$$\frac{v^2}{2} + \frac{a^2}{k - 1} = \frac{w^2}{2} + \frac{{a_0}^2}{k - 1} \tag{31}$$

$$v = w(1 + b_1\beta + b_2\beta^2 + b_3\beta^3 + b_4\beta^4 + \ldots) \tag{32}$$

where

$$b_1 = -(M^2 - 1)^{-\frac{1}{2}}$$

$$b_2 = -(M^2 - 1)^{-2}\left(\frac{1}{2} + \frac{k - 1}{4} M^4\right)$$

$$b_3 = -(M^2 - 1)^{-\frac{7}{2}}\left[\frac{1}{6} + \frac{1}{2} M^2 + \frac{3}{4}(k - 1)M^4 + \frac{3k^2 - 12k + 5}{24} M^6 + \frac{(k + 1)^2}{32} M^8\right]$$

$$b_4 = -(M^2 - 1)^{-5}\left(\frac{1}{24} + \frac{5}{8} M^2 + \frac{-17 + 29k}{24} M^4 + \frac{-1 - 27k + 12k^2}{24} M^6 + \right.$$

$$\left. \frac{5 + 5k - k^2 + k^3}{16} M^8 + \frac{-5 - k - 3k^2 + 3k^3}{48} M^{10}\right)$$

. . . .

$$\frac{\theta}{\theta_0} = 1 + l_3\beta^3 + l_4\beta^4 + \ldots \tag{33}$$

where

$$l_3 = \frac{k(k^2 - 1)}{12} M^6 (M^2 - 1)^{-\frac{3}{2}}$$

$$l_4 = \frac{k(k^2 - 1)}{12} M^6 (M^2 - 1)^{-3} \left[4 + 2(k - 2)M^2 - (k - 1)M^4\right]$$

. . . .

$$\frac{dy}{dx} = e_0 + e_1\beta + e_2\beta^2 + \ldots \tag{34}$$

where

$$e_0 = (M^2 - 1)^{-\frac{1}{2}}$$

$$e_1 = \frac{k + 1}{4} M^4 (M^2 - 1)^{-2}$$

. . . .

Condition 31 shows that, disregarding the presence of jump discontinuities in the functions v, β, ρ, p, a, equation 21, just as in case I, is valid throughout the entire region filled by the flow under consideration. However, condition 22 is not, generally speaking, fulfilled in the case now under consideration. However, there is the possibility of speaking of satisfying this condition approximately. In fact, consider equation 33. Its right-hand side does not contain terms in the first and second powers of β. Therefore, for slightly disturbed flows, equation 22 may be regarded as approximately satisfied on the line OD and consequently throughout the entire region filled by the flow under consideration. From this it follows that in the region of disturbed flow equation 24 may be regarded as approximately satisfied, which means that equations 25 and 26 hold on characteristics. For values of β and v near 0 and w, respectively, equation 28 may be written in the form of a series

$$v = w(1 + b_1'\beta + b_2'\beta^2 + b_3'\beta^3 + b_4'\beta^4 + \ldots) \tag{35}$$

where

$$b_1' = -(M^2 - 1)^{-\frac{1}{2}} = b_1$$

$$b_2' = -(M^2 - 1)^{-2}\left(\frac{1}{2} + \frac{k - 1}{4} M^4\right) = b_2$$

$$b_3' = -(M^2 - 1)^{-\frac{7}{2}}\left[\frac{1}{6} + \frac{1}{2} M^2 + \frac{3}{4}(k - 1)M^4 + \frac{2k^2 - 5k + 3}{12} M^6\right]$$

$$b_4' = -(M^2 - 1)^{-5}\left(\frac{1}{24} + \frac{5}{8} M^2 + \frac{-17 + 29k}{24} M^4 + \frac{3 - 19k + 16k^2}{24} M^6 + \frac{3 - 2k - 5k^2 + 4k^3}{32} M^8 + \frac{-3 + 8k - 7k^2 + 2k^3}{96} M^{10}\right)$$

. . . .

Comparing equations 32 and 35 we see that for slightly disturbed streams the first may be substituted for the second with good approximation. Consequently, for slightly disturbed flows, equation 28 will be approximately satisfied along the line OD. Since, on the other hand, along each characteristic of the first family equation 25 is approximately satisfied, equation 28 will be approximately satisfied throughout the entire region of disturbed flow. The approximate expressions for the functions v, β, ρ, p, a are constituted exactly like the accurate expressions for these functions in case I. Substituting the approximate expression for the function β in the right-hand side of equation 34, we obtain a differential equation of the first degree for the approximate determination of the form of the shock line. Summing up our considerations we can deduce that the accurate results contained in case I can serve as approximate results for case II, and further that expression 29 can serve as an approximate expression for the pressure on the flow around a contour in case II. These same considerations show that there is no sense in calculating all terms in this expression. It is sufficient to limit ourselves to the first two or three terms.

From all that has been said about cases I and II one may conclude that the form of the contour K may be made up in such a manner that artfully constructed shocks may be caused to appear in the region of flow around the fixed frictionless wall. In such cases when we pay attention to this phenomenon, the results we have obtained are valid, not for the entire region of flow around the fixed frictionless wall, but only for that part in the neighborhood of the front side of the flow around a contour. The fundamental problem of the present work is the construction of approximate expressions for the pressure on the flow around a contour in case II, with the calculation of the circulation of the flow occasioned by the presence of the shock discontinuity OD. In spite of the fact that in the case of the presence of circulation it is impossible to integrate equations 15 and 17, there is the possibility, however, of making up such combinations of differentials from equations 14, 15, 16, and 17, adding to these equations expressions for differentials of the stream function, that with the aid of these combinations it is possible to construct expressions which we shall integrate. Investigations concerning the preceding construction constitute the contents of the following section.

PART II

Suppose we have a flow corresponding to case II of the preceeding section. Assume that in this flow the hydrodynamical elements in the region of the disturbed stream differ infinitely little from the hydrodynamical elements in the region of the undisturbed flow. We revamp somewhat our notion of the region of disturbed flow. Shortly before we agreed to apply this name to the region bounded by the curvelinear triangle made up of the curve OC_2 (contour K), the shock line OC_1, and the characteristic of the first family C_1C_2 emerging from the lowest point of the contour K (fig. 4). Taking into consideration equations 5, 14, 18, and 34, it is not difficult to conclude that with the assumptions made just now relative to the hydrodynamical elements the curvelinear triangle OC_1C_2 differs infinitely little from the isoceles straight-line triangle $O'C_1'C_2'$ (fig. 5) where the equal sides $O'C_1'$ and $C_1'C_2'$ are parallel to characteristics of the second and first families in the undisturbed flow. As for the functions $\beta_k(x)$, β, v, ρ, p, a we assume that they all have the properties of differentiability and continuity to as many degrees as may be necessary to insure legitimacy of operations which are performed upon them. Moreover, we assume that in the flow under consideration the infinitesimal quantities $\beta_k(x)$, $\beta_k'(x)$, $\beta_k''(x)$, β, $v - w$, $\rho - \rho_0$, $p - p_0$, $a - a_0$ have the same order of magnitude.

Taking this last group of infinitesimals as fundamental (having unit order of magnitude) we shall agree in what follows to adhere to the following system of notations appearing in investigations involving infinitely small quantities. By ϵ_m (m being any positive integer) let us denote an infinitesimal whose order of magnitude is not less than m. Clearly such a mode of notation does not exclude the possibility of several different infinitesimals being denoted by the same symbol, and, vice versa. The same infinitesimal may be denoted by several different symbols. Thus, for example, if an infinitesimal α is denoted by ϵ_4, the infinitesimal 2α may also be denoted by ϵ_4, and, moreover, the infinitesimals α and 2α may be denoted by ϵ_3, ϵ_2, ϵ_1.

On an arbitrary characteristic of the second or first family the equation

$$d\psi = \rho v(\sin \beta dx - \cos \beta dy) \tag{36}$$

will be satisfied by virtue of equation 8 throughout the entire region filled by the flow. Eliminating dx and dy from equations 14, 15, and 36 and taking into account formulas 13 and 21, we arrive at the equation

$$d(v \cos \beta) + m_2 d(v \sin \beta) = \Phi_1 d \ln \theta \tag{37}$$

which is satisfied on any characteristic of the first family. Here Φ_1 denotes the quantity

$$\Phi_1 = \frac{a^2\left[v^2 \sin \beta \cos \beta - (v^2\cos^2\beta - a^2)m_1\right]}{k(k - 1)v(v^2\cos^2\beta - a^2)(m_1 \cos \beta - \sin \beta)} \tag{38}$$

On the other hand, having the integral 25 of the equation

$$d(v \cos \beta) + m_2 d(v \sin \beta) = 0 \tag{39}$$

it is easy to find an integrating factor L_1 of this equation, such that after multiplying by L_1 it may be written in the form

$$d\left[\beta + \varphi(v)\right] = 0 \tag{40}$$

In order to determine L_1 we have the obvious relationship

$$L_1\left[d(v \cos \beta) + m_2 d(v \sin \beta)\right] = d\left[\beta + \varphi(v)\right] \tag{41}$$

from which we obtain without difficulty

$$L_1(m_2 \cos \beta - \sin \beta) v d\beta = d\beta \tag{42}$$

consequently

$$L_1 = \frac{1}{v(m_2 \cos \beta - \sin \beta)} \tag{43}$$

If now we multiply both sides of equation 37 by L_1, this equation takes the form

$$d\left[\beta + \varphi(v)\right] = H_1 d \ln \theta \tag{44}$$

where H_1 denotes the quantity

$$H_1 = \frac{(v^2 \cos^2 \beta - a^2) m_1 - v^2 \sin \beta \cos \beta}{k(k - 1) v^2} \tag{45}$$

We denote by H_{10} the value of H_1 at $v = w$, $\beta = 0$. We have

$$H_{10} = \frac{1}{k(k - 1)M^2}(M^2 - 1)^{1/2} \tag{46}$$

Equation 44 may be rearranged in the following fashion

$$d\left[\beta + \varphi(v)\right] = H_{10}d \ln \frac{\theta}{\theta_0} + (H_1 - H_{10})d \ln \frac{\theta}{\theta_0} \tag{47}$$

Now choose an arbitrary point S in the region of disturbed flow and lead a characteristic of the first family through it. We denote the point of intersection of this characteristic with the shock line by A (fig. 6). Integrating both sides of equation 47 along the above characteristic from point A to point S we obtain

$$\beta_s + \varphi(v_s) - \beta_a - \varphi(v_a) = H_{10}\left(\ln \frac{\theta_s}{\theta_0} - \ln \frac{\theta_a}{\theta_0}\right) + \int_{AS}(H_1 - H_{10})d \ln \frac{\theta}{\theta_0} \tag{48}$$

where β_s, v_s, θ_s denote, respectively, the values of β, v, θ at the point S and β_a, v_a, θ_a denote the values of these quantities at the point A. Taking account of equation (32) we have

$$v_a = w(1 + b_1\beta_a + b_2\beta_a^2 + b_3\beta_a^3 + b_4\beta_a^4 + \ldots) \tag{49}$$

We introduce the quantity v_{a1} defined with the help of the expansion 35 in the following fashion

$$v_{a1} = w(1 + b_1\beta_a + b_2\beta_a^2 + b_3'\beta_a^3 + b_4'\beta_a^4 + \ldots) \qquad \text{(cf eq. (35) -Tr.)} \tag{50}$$

By this definition of the quantity v_{a1} we have

$$\beta_a + \varphi(v_{a1}) = \varphi(w) \tag{51}$$

With the help of formulas 49, 50, and 51 we rearrange the expression $\beta_a + \varphi(v_a)$ in the following manner

$$\begin{aligned}\beta_a + \varphi(v_a) &= \beta_a + \varphi(v_{a1}) + \varphi(v_a) - \varphi(v_{a1}) \\ &= \varphi(w) + \varphi(v_a) - \varphi(v_{a1}) \\ &= \varphi(w) + \varphi'(w)\Big[(v_a - w) - (v_{a1} - w)\Big] + \\ &\quad \frac{1}{2}\varphi''(w)\Big[(v_a - w)^2 - (v_{a1} - w)^2\Big] + \dots \\ &= \varphi(w) + w\varphi'(w)\Big[(b_3 - b_3')\beta_a^{\,3} + (b_4 - b_4')\beta_a^{\,4}\Big] + \\ &\quad w^2\varphi''(w)b_1(b_3 - b_3')\beta_a^{\,4} + \epsilon_5\end{aligned} \tag{52}$$

Calculating $\varphi'(w)$, $\varphi''(w)$ we obtain

$$\varphi'(w) = -\frac{1}{wb_1} \tag{53}$$

$$\varphi''(w) = \frac{2b_2}{w^2 b_1^{\,3}} \tag{54}$$

Using formulas 52, 53, and 54 we easily find

$$\beta_a + \varphi(v_a) = \varphi(w) - \frac{1}{b_1}(b_3 - b_3')\beta_a^3 + \left[\frac{2b_2}{b_1^2}(b_3 - b_3') - \frac{1}{b_1}(b_4 - b_4')\right]\beta_a^4 + \epsilon_5 \tag{55}$$

Now pass a stream line through the point S and denote by P the point of intersection of this line with the shock line. Since the stream function C is constant along this line we have

$$\theta_s = \theta_p \tag{56}$$

where θ_p denotes the value of θ at the point p. Taking logarithms of both sides of equation (33) we obtain

$$\ln \frac{\theta}{\theta_0} = l_3\beta^3 + l_4'\beta^4 + l_5'\beta^5 + \ldots \tag{57}$$

Since the values of the coefficients l_4', l_5', . . . will not be needed in what follows, we shall not calculate them.

Using formulas (56) and (57) we easily see that

$$\ln \frac{\theta_s}{\theta_0} - \ln \frac{\theta_a}{\theta_0} = l_3(\beta_p^3 - \beta_a^3) + l_4'(\beta_p^4 - \beta_a^4) + \epsilon_5 \tag{58}$$

where β_p denotes the value of β at the point p. Assuming that the mean value theorem is applicable to the integral arising from the right-hand side of equation (48), we easily find[1]

[1] Instead, take a slightly more general assumption admitting the part AS of the characteristic under consideration to be divided in the same finite number of parts in such manner that on each part the mean value theorem can be applied to the integral under investigation.

$$\int_{AS} (H_1 - H_{10}) d \ln \frac{\theta}{\theta_0} = l_3(\tilde{H}_1 - H_{10})(\beta_p{}^3 - \beta_a{}^3) + \epsilon_5 \tag{59}$$

(In consequence of this equation one must keep in mind that $\tilde{H}_1 - H_{10} = \epsilon_1$). Here $\tilde{H}_1$ denotes the value of H_1 at some point on the characteristic under consideration between the points A and S.

Using relations (55), (58), and (59) we write equation (48) in the following form

$$\beta_s + \varphi(v_s) = \varphi(w) - \frac{1}{b_1}(b_3 - b_3')\beta_a{}^3 + \left[\frac{2b_2}{b_1{}^2}(b_3 - b_3') - \frac{1}{b_1}(b_4 - b_4')\right]\beta_a{}^4 +$$

$$H_{10}l_3(\beta_p{}^3 - \beta_a{}^3) + H_{10}l_4'(\beta_p{}^4 - \beta_a{}^4) +$$

$$l_3(\tilde{H}_1 - H_{10})(\beta_p{}^3 - \beta_a{}^3) + \epsilon_5 \tag{60}$$

We denote by B the intersection of the characteristic of the first family under consideration with the contour K. Applying formula (60) to the point B (which is possible, since the point S was chosen arbitrarily) we obtain

$$\beta_b + \varphi(v_b) = \varphi(w) - \frac{1}{b_1}(b_3 - b_3')\beta_a{}^3 + \left[\frac{2b_2}{b_1{}^2}(b_3 - b_3') - \frac{1}{b_1}(b_4 - b_4')\right]\beta_a{}^4 +$$

$$H_{10}l_3(\beta_0{}^3 - \beta_a{}^3) + H_{10}l_4'(\beta_0{}^4 - \beta_a{}^4) +$$

$$l_3(\tilde{H}_1 - H_{10})(\beta_0{}^3 - \beta_a{}^3) + \epsilon_5 \tag{61}$$

where β_b, v_b denote, respectively, the values of β and v at the point B and β_0 denotes the value of β at the point O.

We now proceed to the derivation of an expression for β_a. From formula (60) we have

$$\beta_s + \varphi(v_s) = \varphi(w) + \epsilon_3 \tag{62}$$

From equation (62), using formula (35) we obtain

$$v_s = w(1 + b_1\beta_s + b_2{\beta_s}^2) + \epsilon_3 \tag{63}$$

and denoting by m_{2s} the value of m_2 at the point S we obtain, by using formulas (19) and (63)

$$m_{2s} = (M^2 - 1)^{-1/2} + \frac{k + 1}{2} M^4(M^2 - 1)^{-2} \beta_s + \epsilon_2$$

$$= e_0 + 2e_1\beta_s + \epsilon_2 \tag{64}$$

Analagous to the derivation of equation (47), which holds on characteristics of the first family, we may derive equation

$$d\left[\beta - \varphi(v)\right] = H_{20}d \ln \frac{\theta}{\theta_0} + (H_2 - H_{20})d \ln \frac{\theta}{\theta_0} \tag{65}$$

which is valid on characteristics of the second family. Here H_2 denotes the function defined as

$$H_2 = \frac{(v^2\cos^2\beta - a^2)m_2 - v^2\sin \beta \cos \beta}{k(k - 1)v^2} \tag{66}$$

and H_{20} denotes the value of this function at $\beta = 0$, $v = w$.

Now pass a characteristic of the second family through the point S and denote by Q its intersection with the contour K. Integrating both sides of equation (65) along this characteristic from the point Q to the point S we obtain

$$\beta_s - \beta_q - \left[\varphi(v_s) - \varphi(v_q)\right] = H_{20}\left(\ln \frac{\theta_s}{\theta_0} - \ln \frac{\theta_q}{\theta_0}\right) + \int_{QS} (H_2 - H_{20}) d \ln \frac{\theta}{\theta_0} \tag{67}$$

where β_q, v_q, θ_q denote respective the values of β, v, θ at the point Q. Since the contour K is a stream line we have

$$\theta_q = \theta^{(0)} \tag{68}$$

where $\theta^{(0)}$ denotes the value of θ at the point O. Assuming that the mean value theorem can be applied to the integral arising from the right-hand side of equation (67) we easily find, with the aid of formulas (56), (57), and (68)

$$\beta_s - \beta_q - \left[\varphi(v_s) - \varphi(v_q)\right] = \epsilon_3 \tag{69}$$

Applying formula (62) at the point Q we have

$$\beta_q + \varphi(v_q) = \varphi(w) + \epsilon_3 \tag{70}$$

Eliminating $\varphi(w)$ from equations (62) and (70) we arrive at the following equation

$$\beta_s - \beta_q + \left|\varphi(v_s) - \varphi(v_q)\right| = \epsilon_3 \tag{71}$$

Families (69) and (71) give

$$\beta_s = \beta_q + \epsilon_3 \tag{72}$$

On the shock line we take an arbitrary point F (fig. 7) and pass through it a characteristic of the second family in the region of the disturbed flow and we denote by β_f, m_{2f}, respectively, the values β and m_2 at the point F. Applying formula (64) at the point F we obtain

$$m_{2f} = e_0 + 2e_1\beta_f + \epsilon_2 \tag{73}$$

We denote by $\left(\frac{dy}{dx}\right)_f$ the slope of the tangent to the shock line at the point F. From equations (34) we have

$$\frac{dy}{dx}_f = e_0 + e_1\beta_f + \epsilon_2 \tag{74}$$

Comparing formulas (73) and (74) we see that the characteristic of the second family passing through F and the shock line at this intersection make an infinitesimal angle with each other moreover, if

$$\beta_f > 0 \tag{75}$$

the slope of the characteristic of the second family is greater than the slope of the shock line at the point F.[2]

[2]It is easy to show that if the shock line is unbroken and moreover condition (30) is satisfied the inequality $\beta < 0$ is impossible on this line. As a matter of fact, in the opposite case the shock line is broken since with $\beta < 0$ condition (34) must be replaced by the following condition in virtue of Tsemplen's theorem

$$\frac{dy}{dx} = -(e_0 - e_1\beta + e_2\beta^2 - \ . \ . \ .)$$

Denoting by L the intersection of the characteristic of the second family under consideration with the contour K and by x_l abscissa of this point, we have

$$x_l = \epsilon_1 \tag{76}$$

Let β_l denote the value of β at the point L. Using equations (5) and (76) and MacLauren's formula we obtain

$$\beta_l = \beta_k(0) + \beta_k'(0)x_l + \epsilon_3 \tag{77}$$

Applying formula (72) at the point F we obtain

$$\beta_f = \beta_l + \epsilon_3 \tag{78}$$

As a consequence of equations (77) and (78)

$$\beta_f = \beta_k(0) + \beta_k'(0)x_l + \epsilon_3 \tag{79}$$

Since the point F was chosen arbitrarily on the shock line by use of equations (74), (76), and (79) we can obtain the following differential equation for the shock line

$$\frac{dy}{dx} = e_0 + e_1\beta_k(0) + \epsilon_2 \tag{80}$$

Consequently the equation of the shock line may be written in the following form

$$y = \left[e_0 + e_1\beta_k(0)\right]x + \epsilon_2 \tag{81}$$

Applying formulas (64) and (72) to an arbitrary point situated on the characteristic of the second family FL we easily obtain the differential equation of this line from the following form

$$\frac{dy}{dx} = e_0 + 2e_1\beta_l + \epsilon_2 \tag{82}$$

Employing formulas (76) and (77) this equation may be written

$$\frac{dy}{dx} = e_0 + 2e_1\beta_k(0) + \epsilon_2 \tag{83}$$

Consequently the equation of the characteristic FL may be written in the form

$$y = y_l + \left[e_0 + 2e_1\beta_k(0)\right](x - x_l) + \epsilon_2 \tag{84}$$

where y_l denotes the ordinate of the point L.

On the other hand, taking account of formulas (3) and (76) we have

$$y_l = \int_0^{x_l} \tan \beta_k(x)dx = \epsilon_2 \tag{85}$$

Employing formulas (85) and (76) we may write equation (84) in the form

$$y = \left[e_0 + 2e_1\beta_k(0)\right]x - e_0x_l + \epsilon_2 \tag{86}$$

Applying formulas (81) and (86) at the point F and denoting by x_f, y_f the coordinates of this point we obtain

$$\left.\begin{aligned} y_f &= \left[e_0 + e_1\beta_k(0)\right]x_f + \epsilon_2 \\ y_f &= \left[e_0 + 2e_1\beta_k(0)\right]x_f - e_0 x_l + \epsilon_2 \end{aligned}\right\} \tag{87}$$

From equation (87) we easily obtain

$$x_l = \frac{e_1}{e_0} x_f \beta_k(0) + \epsilon_2 \tag{88}$$

Replacing x_l in the right-hand side of equation (79) by the expression in formula (88) we obtain

$$\beta_f = \beta_k(0) + \frac{e_1}{e_0} x_f \beta_k(0)\beta_k'(0) + \epsilon_3 \tag{89}$$

We denote by x_a, y_a the coordinates of the point A and by x_b, y_b the coordinates of the point B. Applying formula (89) at the point A we arrive at the following result

$$\beta_a = \beta_k(0) + \frac{e_1}{e_0} x_a \beta_k(0)\beta_k'(0) + \epsilon_3 \tag{90}$$

We now express x_a in terms of x_b. To this end, using formulas (14) and (18), we write the differential equation for characteristics of the second family in the following fashion

$$\frac{dy}{dx} = -e_0 + \epsilon_1 \tag{91}$$

Employing formula (91) we write the equation for the characteristic AB of the first family in the form

$$y = y_b - e_0(x - x_b) + \epsilon_1 \tag{92}$$

Taking account of formula (3) we have

$$y_b = \int_0^{x_b} \tan \beta_k(x)dx = \epsilon_1 \tag{93}$$

consequently equation (92) may be written

$$y = -e_0(x - x_b) + \epsilon_1 \tag{94}$$

On the other hand, equation (81) for the shock line may be written in the form

$$y = e_0 x + \epsilon_1 \tag{95}$$

and applying formulas (94) and (95) at the point A we obtain

$$\left.\begin{aligned} y_a &= -e_0(x_a - x_b) + \epsilon_1 \\ y_a &= e_0 x_a + \epsilon_1 \end{aligned}\right\} \tag{96}$$

From equation (96) we easily find

$$x_a = \frac{x_b}{2} + \epsilon_1 \tag{97}$$

And consequently

$$\beta_a = \beta_k(0) + \frac{e_1}{2e_0} x_b \beta_k(0)\beta_k'(0) + \epsilon_3 \tag{98}$$

Substituting this expression for β_a in the right-hand side of equation (61) and substituting β_0 for $\beta_k(0)$ in the fundamental formula (5) we obtain

$$\begin{aligned}\beta_b + \varphi(v_b) = \varphi(w) - \frac{1}{b_1}(b_3 - b_3')\beta_k^3(0) - \\ \left[\frac{1}{b_1}(b_4 - b_4') - \frac{2b_2}{b_1^2}(b_3 - b_3')\right]\beta_k^4(0) - \\ \frac{3e_1}{2e_0}\left[\frac{b_3 - b_3'}{b_1} + H_{10}l_3\right]x_b\beta_k^3(0)\beta_k'(0) + \epsilon_5\end{aligned} \tag{99}$$

Employing relation (35), we easily obtain from equation (99)

$$\begin{aligned}v_b = w\Bigg\{1 + b_1\beta_b + b_2\beta_b^2 + b_3'\beta_b^3 + b_4'\beta_b^4 + (b_3 - b_3')\beta_k^3(0) + \\ \left[b_4 - b_4' - \frac{2b_2}{b_1}(b_3 - b_3')\right]\beta_k^4(0) + \frac{2b_2}{b_1}(b_3 - b_3')\beta_k^3(0)\beta_b + \\ \frac{3e_1}{2e_0}\left[b_3 - b_3' + H_{10}l_3b_1\right]x_b\beta_k^3(0)\beta_k'(0)\Bigg\} + \epsilon_5\end{aligned} \tag{100}$$

Substituting v_b, β_b, x_b for v, $\beta_k(x)$ and x, respectively, in formula (100) we arrive at the following final expression for the velocity on the contour K:

$$v = w\left\{1 + b_1\beta_k(x) + b_2\beta_k^{\,2}(x) + b_3{}'\beta_k^{\,3}(x) + b_4{}'\beta_k^{\,4}(x) + (b_3 - b_3{}')\beta_k^{\,3}(0) + \left[b_4 - b_4{}' - \frac{2b_2}{b_1}(b_3 - b_3{}')\right]\beta_k^{\,4}(0) + \frac{2b_2}{b_1}(b_3 - b_3{}')\beta_k^{\,3}(0)\beta_k(x) + \frac{3e_1}{2e_0}\left[b_3 - b_3{}' + H_{10}l_3b_1\right]x\beta_k^{\,3}(0)\beta_k{}'(0)\right\} + \epsilon_5 \tag{101}$$

We now proceed to the derivation of formulas from which the pressure on the contour K can be calculated. Clearly

$$a_0^{\,2} = k\,\frac{p_0}{\rho_0} \tag{102}$$

and moreover, on the contour K the following equation holds

$$\frac{p}{\rho^k} = \theta^{(0)} \tag{103}$$

Employing formulas (7), (10), (21), (23), (102), and (103) we easily obtain the following expression for the pressure on the contour K

$$p = p_0\left[\frac{\theta^{(0)}}{\theta_0}\right]^{-\frac{1}{k-1}}\left[1 - \frac{k-1}{2}M^2\left(\frac{v^2}{w^2} - 1\right)\right]^{\frac{k}{k-1}} \tag{104}$$

On the other hand, by virtue of equations (5) and (33) the following equation holds

$$\frac{\theta(0)}{\theta_0} = 1 + l_3\beta_k{}^3(0) + l_4\beta_k{}^4(0) + \ldots \tag{105}$$

Substituting the expressions for v and $\frac{\theta(0)}{\theta_0}$ obtained in formulas (101) and (105), respectively, in the right-hand side of equation (104) we obtain after elementary transformations the desired formula for the calculation of the pressure on the contour K

$$p = p_0 + q\Big[a_1\beta_k(x) + a_2\beta_k{}^2(x) + a_3\beta_k{}^3(x) + a_4\beta_k{}^4(x) + a_{1d}\beta_k{}^3(0) + a_{2d}\beta_k{}^4(0) + a_{3d}\beta_k{}^3(0)\beta_k(x) + a_{4d}\beta_k{}^3(0)\beta_k{}'(0)x\Big] + \epsilon_5 \tag{106}$$

where

$$a_{1d} = -2(b_3 - b_3{}') - \frac{2l_3}{k(k - 1)M^2}$$

$$= \frac{k + 1}{2} M^4(M^2 - 1)^{-\frac{7}{2}}\left(-\frac{1}{3} + \frac{3 - k}{6} M^2 + \frac{3k - 5}{24} M^4\right)$$

$$a_{2d} = -2(b_4 - b_4{}') + \frac{4b_2}{b_1}(b_3 - b_3{}') - \frac{2l_4}{k(k - 1)M^2}$$

$$= M^4(M^2 - 1)^{-5}\left(-\frac{k + 1}{2} + \frac{5 + 3k - 2k^2}{4} M^2 + \frac{-10 - 3k + 6k^2 - k^3}{8} M^4 + \frac{9 - 7k^2 + 2k^3}{16} M^6 + \frac{-3 + k + 3k^2 - k^3}{32} M^8\right)$$

$$a_{3d} = (b_3 - b_3')\left(2M^2 b_1 - 2b_1 - \frac{4b_2}{b_1}\right) + \frac{2\imath_3 b_1}{k - 1}$$

$$= M^6(M^2 - 1)^{-5}\left(- \frac{k + 1}{6} + \frac{7 + 2k - 5k^2}{24} M^2 + \frac{-4 + 3k + 6k^2 - k^3}{24} M^4 + \frac{3 - 7k - 7k^2 + 3k^3}{96} M^6\right)$$

$$a_{4d} = - \frac{3e_1}{e_0}(b_3 - b_3' + H_{10}\imath_3 b_1)$$

$$= \frac{(k + 1)^2}{16} M^8(M^2 - 1)^{-5}\left(-1 + \frac{3 - k}{2} M^2 + \frac{3k - 5}{8} M^4\right)$$

For $x = 0$ the formula (106) takes the form

$$p = p_0 + q\left[a_1\beta_k(0) + a_2\beta_k^{\,2}(0) + a_3'\beta_k^{\,3}(0) + a_4'\beta_k^{\,4}(0)\right] + \epsilon_5 \qquad (107)$$

where $a_3' = a_3 + a_{1d}$, $a_4' = a_4 + a_{2d} + a_{3d}$.

Formula (107) may be used for the calculation of the pressure on a flat plate which is inclined at an angle $\beta_k(0)$ to the undisturbed flow.

In order to single out of the right-hand side of equation (106) those terms which depend exclusively on the presence of the shock in front of the contour K, we add to the contour K under consideration an arc O'O of finite length in such a manner that this arc is tangent to K at the point O and is parallel to the x-axis at O' (fig. 8). Since the flow around such an additional contour is accomplished without the appearance of shocks (we suppose that the angle between the direction of flow and the x-axis and the derivative of this angle with respect to x are both infinitely small), formula (29) may be employed in the calculation of the pressure on this contour. Comparing formulas (29) and (106) and denoting by Δp_{stoss} the pressure resulting from the presence of the shock front, we obtain

$$\Delta p_{stoss} = q\Big[a_{1d}\beta_k^3(0) + a_{2d}\beta_k^4(0) + a_{3d}\beta_k^3(0)\beta_k(x) +$$

$$a_{4d}\beta_k^3(0)\beta_k'(0)x\Big] + \epsilon_5 \tag{108}$$

We may, in turn single out of the expression for Δp_{stoss} the term depending solely on the vorticity caused by the presence of the shock. In order to do this we add to the contour $K(OC_2)$ a straight-line segment tangent at the point O (segment O'O in fig. 9). With the contour $O'OC_2$ a shock is formed, but the shock line $O'C_1$ is straight so that vortex formation is absent. Calculating the pressure on the portion OC_2 of the contour $O'OC_2$, we obtain

$$p = p_0 + q\Big[a_1\beta_k(x) + a_2\beta_k^2(x) + a_3\beta_k^3(x) + a_4\beta_k^4(x) +$$

$$a_{1d}\beta_k^3(0) + a_{2d}\beta_k^4(0) + a_{3d}\beta_k^3(0)\beta_k(x)\Big] + \epsilon_5 \tag{109}$$

Comparing formulas (106) and (109) and denoting by Δp_{rot} the pressure due to vortex formation caused by the shock, we obtain

$$\Delta p_{rot} = qa_{4d}\beta_k^3(0)\beta_k'(0)x + \epsilon_5 \tag{110}$$

PART III

We now apply the results obtained to the calculation of the lifting force and head resistance of a flat wing with sharp front and rear edges placed in a supersonic stream having constant hydrodynamical elements.

We place the origin O at the front edge of the wing and arrange the coordinate system so that the positive x-axis corresponds to the direction of the velocity of the undisturbed flow and measure angles in the manner used heretofore. Segment OC_2 connecting the front and rear edges (fig. 10) will be called the chord of the wing as in the theory of wings. The length of this curve will be denoted by T and the angle it makes with the x-axis by $\bar{\beta}$.

The form of the wings we are investigating is defined by a pair of contours like that investigated in the preceding section, possessing a pair of common points O, C_2. Comparing ordinates of points on these contours having the same abscissa, we call the upper contour K_u that contour of which every point on the ordinate is greater than the corresponding point on the ordinate of the other contour, which we call the lower contour K_l. The function $\beta_k(x)$ for the upper contour we denote by $\beta_{ku}(x)$ and for the lower by $\beta_{kl}(x)$.

We choose an arbitrary point A on the chord of the wing and denote the distance OA by t. Through A we pass a straight line perpendicular to the chord of the wing and denote by A_u and A_l, respectively, the intersections of this straight line with the upper and lower contours. With the point A_u we associate a unit tangent vector t_u and at the point A_l a unit tangent vector t_l. The vectors t_u and t_l will be directed in such a manner that their projections on the direction OC_2 are positive. We denote by β_u and β_l, respectively, the angles these vectors make with the vector $\overrightarrow{OC_2}$. Clearly β_u, β_l may be regarded as functions of t. We denote by β_{uO} and β_{lO} the values of β_u and β_l at the point O and the values of the derivatives of β_u and β_l with respect to t at the point O by β_{uO}' and β_{lO}'.

The abscissa x of A_u may be calculated from the formula

$$x = t \cos \bar{\beta} - \sin \bar{\beta} \int_0^t \tan \beta_u dt \qquad (111)$$

and that of the point A_l from the formula

$$x = t \cos \bar{\beta} - \sin \bar{\beta} \int_0^t \tan \beta_l dt \qquad (112)$$

The value of the functions $\beta_{ku}(x)$ at the point A_u is determined by the relation

$$\beta_{ku}(x) = \bar{\beta} + \beta_u \qquad (113)$$

and the value of the function $\beta_{kl}(x)$ at the point A_l by the relation

$$\beta_{kl}(x) = \bar{\beta} + \beta_l \tag{114}$$

Moreover we have the relations

$$\int_0^T \tan \beta_u dt = 0 \tag{115}$$

$$\int_0^T \tan \beta_l dt = 0 \tag{116}$$

We assume that $\bar{\beta}$ and also β_u and β_l and other derivatives with respect to t are infinitesimal quantities. From equations (115) and (116) we easily obtain

$$\int_0^T \beta_u dt = -\frac{1}{3}\int_0^T \beta_u{}^3 dt + \epsilon_5 \tag{117}$$

$$\int_0^T \beta_l dt = -\frac{1}{3}\int_0^T \beta_l{}^3 dt + \epsilon_5 \tag{118}$$

Proceeding now to the calculation of the lifting force and head resistance of the wing under consideration, we remark that on the top side of the wing a shock appears when and only when

$$\bar{\beta} + \beta_{u0} > 0 \tag{119}$$

and at the bottom side when and only when

$$\bar{\beta} + \beta_{l0} < 0 \tag{120}$$

We introduce the quantities a_{1u}, a_{2u}, a_{3u}, a_{4u} defined as follows

$$\left.\begin{aligned} a_{1u} &= a_{1d} \\ a_{2u} &= a_{2d} \\ a_{3u} &= a_{3d} \\ a_{4u} &= a_{4d} \end{aligned} \quad \text{if} \quad \bar{\beta} + \beta_{u0} > 0 \right\} \tag{121}$$

$$a_{1u} = a_{2u} = a_{3u} = a_{4u} = 0 \qquad \text{if} \qquad \bar{\beta} + \beta_{l0} \leq 0 \tag{122}$$

In an analogous way we define the quantities a_{1l}, a_{2l}, a_{3l}, a_{4l}

$$\left.\begin{aligned} a_{1l} &= a_{1d} \\ a_{2l} &= a_{2d} \\ a_{3l} &= a_{3d} \\ a_{4l} &= a_{4d} \end{aligned} \quad \text{if} \quad \bar{\beta} + \beta_{l0} < 0 \right\} \tag{123}$$

$$a_{1l} = a_{2l} = a_{3l} = a_{4l} = 0 \qquad \text{if} \qquad \bar{\beta} + \beta_{l0} \geq 0 \tag{124}$$

Denoting by p_u the pressure on the upper contour K_u and by p_l the pressure on the lower contour K_l we easily obtain, with the help of formulas (106), (121), (122), (123), (124)

$$\begin{aligned} p_u = p_0 + q\Big[& a_1\beta_{ku}(x) + a_2\beta_{ku}^2(x) + a_3\beta_{ku}^3(x) + a_4\beta_{ku}^4(x) + \\ & a_{1u}\beta_{ku}^3(0) + a_{2u}\beta_{ku}^4(0) + a_{3u}\beta_{ku}^3(0)\beta_{ku}(x) + a_{4u}\beta_{ku}^3(0)\beta_{ku}'(0)x\Big] + \epsilon_5 \end{aligned} \tag{125}$$

$$p_l = p_0 + q\Big[-a_1\beta_{kl}(x) + a_2\beta_{kl}^2(x) - a_3\beta_{kl}^3(x) + a_4\beta_{kl}^4(x) -$$

$$a_{1l}\beta_{kl}^3(0) + a_{2l}\beta_{kl}^4(0) + a_{3l}\beta_{kl}^3(0)\beta_{kl}(x) + a_{4l}\beta_{kl}^3(0)\beta_{kl}'(0)x\Big] + \epsilon_5 \tag{126}$$

Let $\vec{P}$ denote the resultant vector of the hydrodynamical force acting on a unit length of the wing under consideration. We have then

$$\vec{P} = \oint p\vec{n}ds \tag{127}$$

where $\vec{n}$ denotes a unit vector normal to the contour of the wing and directed inwards.

We introduce the dimensionless coefficient of the lifting force C_y and the dimensionless coefficient of head resistance C_x. These coefficients are defined by the formulas

$$C_y = \frac{P_y}{qT} \tag{128}$$

$$C_x = \frac{P_x}{qT} \tag{129}$$

where P_y and P_x denote the projections of the vector $\vec{P}$ on the x- and y-axes, respectively. From formulas (127), (128), (129) we have

$$C_y = -\frac{1}{qT}\int_0^l p_u dx + \frac{1}{qT}\int_0^l p_l dx \tag{130}$$

$$C_x = + \frac{1}{qT} \int_0^l p_u \tan \beta_{ku}(x)dx - \frac{1}{qT} \int_0^l p_l \tan \beta_{kl}(x)dx \tag{131}$$

where l denotes the abscissa of the point C_2. With the aid of formulas (111), (112), (113), (114), (117), (118), (125), (126), (130), and (131) after a few elementary transformations we obtain

$$C_y = C_{y1} + C_{y2} + C_{y3} + C_{y4} + \epsilon_5 \tag{132}$$

where

$$C_{y1} = -2a_1\bar{\beta}$$

$$C_{y2} = -\frac{a_2}{T} \int_0^T \left(\beta_u{}^2 - \beta_l{}^2\right)dt$$

$$C_{y3} = (a_1 - 2a_3)\bar{\beta}^3 - a_{1u}(\bar{\beta} + \beta_{u0})^3 - a_{1l}(\bar{\beta} + \beta_{l0})^3 +$$

$$\frac{1}{T}(a_1 - 3a_3)\bar{\beta} \int_0^T \left(\beta_u{}^2 + \beta_l{}^2\right)dt + \frac{1}{T}\left(\frac{a_1}{3} - a_3\right) \int_0^T \left(\beta_u{}^3 + \beta_l{}^3\right)dt$$

$$C_{y4} = -a_{2u}(\bar{\beta} + \beta_{u0})^4 + a_{2l}(\bar{\beta} + \beta_{l0})^4 - a_{3u}\bar{\beta}(\bar{\beta} + \beta_{u0})^3 +$$

$$a_{3l}\bar{\beta}(\bar{\beta} + \beta_{l0})^3 - \frac{T}{2} a_{4u}(\bar{\beta} + \beta_{u0})^3\beta_{u0}' + \frac{T}{2} a_{4l}(\bar{\beta} + \beta_{l0})^3\beta_{l0}' +$$

$$\frac{1}{T}\left(\frac{5}{2} a_2 - 6a_4\right)\bar{\beta}^2 \int_0^T(\beta_u{}^2 - \beta_l{}^2)dt + \frac{1}{T}\left(\frac{5}{3} a_2 - 4a_4\right)\bar{\beta} \int_0^T (\beta_u{}^3 - \beta_l{}^3)dt -$$

$$\frac{a_4}{T} \int_0^T \left(\beta_u{}^4 - \beta_l{}^4\right)dt$$

$$C_x = C_{x2} + C_{x3} + C_{x4} + \epsilon_5 \tag{133}$$

where

$$C_{x2} = 2a_1\bar{\beta}^2 + \frac{a_1}{T}\int_0^T \left(\beta_u{}^2 + \beta_l{}^2\right)dt$$

$$C_{x3} = \frac{3a_2\bar{\beta}}{T}\int_0^T \left(\beta_u{}^2 - \beta_l{}^2\right)dt + \frac{a_2}{T}\int_0^T \left(\beta_u{}^3 - \beta_l{}^3\right)dt$$

$$C_{x4} = \left(2a_3 - \frac{a_1}{3}\right)\bar{\beta}^4 + a_{1u}\bar{\beta}(\bar{\beta} + \beta_{u0})^3 + a_{1l}\bar{\beta}(\bar{\beta} + \beta_{l0})^3 +$$
$$\frac{1}{T}\left(6a_3 - \frac{a_1}{2}\right)\bar{\beta}^2 \int_0^T \left(\beta_u{}^2 + \beta_l{}^2\right)dt + \frac{1}{T}\left(4a_3 - \frac{a_1}{3}\right)\bar{\beta}\int_0^T \left(\beta_u{}^3 + \beta_l{}^3\right)dt +$$
$$\frac{1}{T}\left(a_3 + \frac{a_1}{3}\right)\int_0^T \left(\beta_u{}^4 + \beta_l{}^4\right)dt$$

Let us consider a numerical example.

Suppose

$$k = 1.405, \qquad M = 1.5, \qquad p_0 = 1.033 \text{ kg/cm}^2 \tag{134}$$

Then

$$q = \frac{p_0 k M^2}{2} = 1.633 \text{ kg/cm}^2$$

$$\left.\begin{aligned}
a_1 &= 2(M^2 - 1)^{-1/2} = 1.789 \\
a_2 &= (M^2 - 1)^{-2}(2 - 2M^2 + 1.203M^4) = 2.296 \\
a_3 &= (M^2 - 1)^{-7/2}(1.333 - 2M^2 + 4.008M^4 - 1.815M^6 + \\
&\quad 0.4008M^8) = 3.082 \\
a_4 &= (M^2 - 1)^{-5}(0.6667 - 0.6667M^2 + 5.616M^4 - 3.824M^6 + \\
&\quad 2.969M^8 - 0.7840M^{10} + 0.07992M^{12}) = 8.290
\end{aligned}\right\} \tag{135}$$

(Equations continued on next page).

$$\left.\begin{aligned}
a_{1d} &= 1.203M^4(M^2 - 1)^{-7/2}(-0.3333 + 0.2658M^2 - 0.03271M^4) = 0.2766 \\
a_{2d} &= M^4(M^2 - 1)^{-5}(-1.203 + 1.317M^2 - 0.6431M^4 + 0.04556M^6 + \\
&\qquad 0.04855M^8) = 0.4448 \\
a_{3d} &= M^6(M^2 - 1)^{-5}(-0.4008 - 0.0025M^2 + 0.3869M^4 - 0.1285M^6) = 0.3318 \\
a_{4d} &= 0.3615M^8(M^2 - 1)^{-5}(-1 + 0.7975M^2 - 0.09813M^4) = 0.9035
\end{aligned}\right\} \begin{pmatrix}135\\ \text{conc.}\end{pmatrix}$$

Let us take as the functions β_u, β_l

$$\left.\begin{aligned}
\beta_u &= -2\bar{\beta} + \frac{4\bar{\beta}}{T}t \\
\beta_l &= 0
\end{aligned}\right\} \tag{136}$$

Moreover we assume that

$$\bar{\beta} < 0 \tag{137}$$

The form and position of the profile of the wing, determined by equations (136) and condition (137), is shown in figure 11. It is easily seen that the straight line S_1S_2 drawn perpendicular to the wing through its mid point is the axis of symmetry of the profile under consideration. From equations (136) and condition (137) we have

$$\left.\begin{aligned}
\bar{\beta} + \beta_{u0} &= -\bar{\beta} > 0 \\
\bar{\beta} + \beta_{l0} &= \bar{\beta} < 0
\end{aligned}\right\} \tag{138}$$

Consequently,

$$\left.\begin{aligned}
a_{1u} = a_{1l} = a_{1d} = 0.2766 \\
a_{2u} = a_{2l} = a_{2d} = 0.4448 \\
a_{3u} = a_{3l} = a_{3d} = 0.3318 \\
a_{4u} = a_{4l} = a_{4d} = 0.9035
\end{aligned}\right\} \qquad (139)$$

Using formulas (132), (133), (135), (136), and (139) we obtain

$$\left.\begin{aligned}
C_y &= -2a_1\bar{\beta} - \frac{4}{3} a_2\bar{\beta}^2 + \left(\frac{7}{3} a_1 - 6a_3\right)\bar{\beta}^3 + \\
&\quad \left(\frac{10}{3} a_2 - \frac{56}{5} a_4 + 2a_{3d} + 2a_{4d}\right)\bar{\beta}^4 + \epsilon_5 \\
&= -3.578\bar{\beta} - 3.061\bar{\beta}^2 - 14.32\bar{\beta}^3 - 82.73\bar{\beta}^4 + \ldots \\
C_x &= \frac{10}{3} a_1\bar{\beta}^2 + 4a_2\bar{\beta}^3 + \left(\frac{a_1}{15} + \frac{66}{5} a_3\right)\bar{\beta}^4 + \epsilon_5 \\
&= 5.963\bar{\beta}^2 + 9.184\bar{\beta}^3 + 40.8\bar{\beta}^4 + \ldots
\end{aligned}\right\} \qquad (140)$$

Let

$$\left.\begin{aligned}
\bar{\beta} &= -\frac{\pi}{36} \\
T &= 100 \text{ cm}
\end{aligned}\right\} \qquad (141)$$

Then

$$\left.\begin{aligned} C_y &= 0.2936 \\ C_x &= 0.04168 \end{aligned}\right\} \tag{142}$$

Knowing C_y, C_x, T, and q, we easily obtain

$$\left.\begin{aligned} P_y &= qTC_y = 47.9 \text{ kg/cm} \\ P_x &= qTC_x = 6.81 \text{ kg/cm} \end{aligned}\right\} \tag{143}$$

SUMMARY

In the present work the problem of a flow of stream of ideal gas around a thin wing at small angles of attack is investigated, this stream being supposed to be two-dimensional, stationary, supersonic and deprived of heat-conduction.

In the initial part of the work, the problem is stated, and the well-known results obtained by Ackeret, Prandtl, and Busemann are cited. These results, as known, are obtained on the basis of the potential supersonic streams theory, which is founded on the existence of integrable combinations of characteristics of differential equations concerning this problem, and in which some peculiarities of the dynamical conditions on the line of the shock wave are utilized.

In the second part the approximate solution of the problem is given with an allowance for vortex-formation caused by the change of entropy along the shock wave, when receding from the leading edge of the wing, near which this shock wave is formed. For this purpose differential equations of characteristics non admitting integrable combinations are to be dealt with. The solution is obtained by means of a special method, which enables us to find the approximate integrable combinations of differential equations of the characteristics. The obtained combinations let us receive the approximate formula of pressure in any point of the contour of the wing investigated. From this formula the term is easily segregated depending exclusively on the vortex formulation, caused by

the change of entropy along the shock wave. The characteristic distinction of this term of the obtained formula of pressure from the other ones, is that it includes the curvature of the wing contour at the leading edge and the distance from this edge up to the element of the wing for which the pressure is calculated.

In the third part of the work the expressions for lift and drag coefficients of the wing are given, on the base of the formula of pressure obtained above. In conclusion a numerical example is studied.

Translated by R. Shaw
Institute of Mathematical Sciences

REFERENCES

1. Meyer, Th.: Über zweidimensionale Bewegungsvorgänge in einem Gas, das mit Überschallgeschwindigkeit strömt. Forsch.-Arb., Ing.-Wes., 62, 1908.

2. Prandtl, L., and Busemann, A.: Nährungsverfahren sur zeichnerischen Ermittlung von ebenen Strömungen mit Überschallgeschwindigkeiten. Stodola Festschrift, Zürich 1929.

3. Ackeret, I.: Gasdynamik. Handbuch der Physik, 19, B. VII.

4. Busemann, A.: Aerodynamischer Auftrieb bei Überschallgeschwindigkeit. Luftfahrtforschung, 1935.

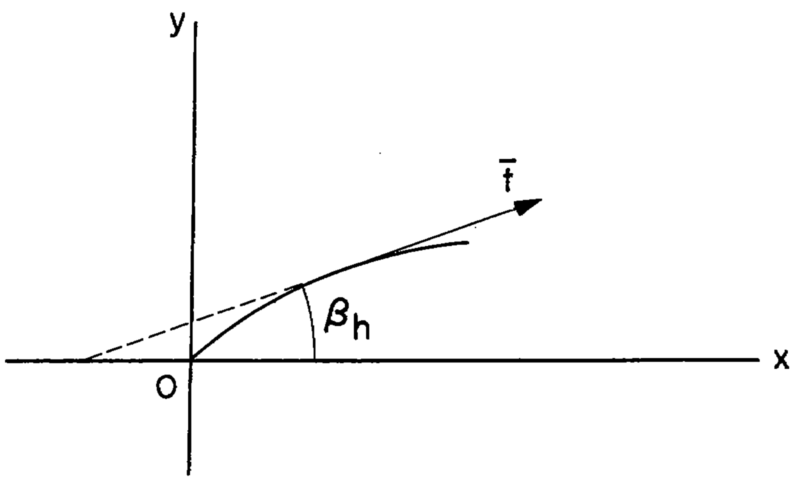

Figure 1.

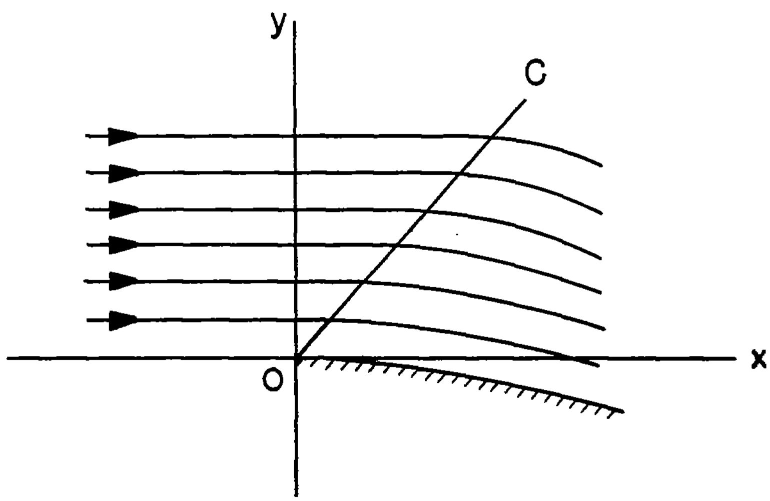

Figure 2(a).

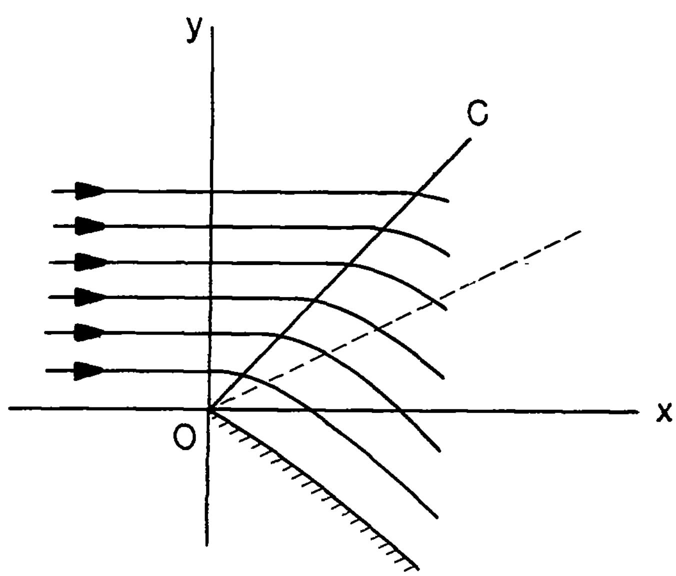

Figure 2(b).

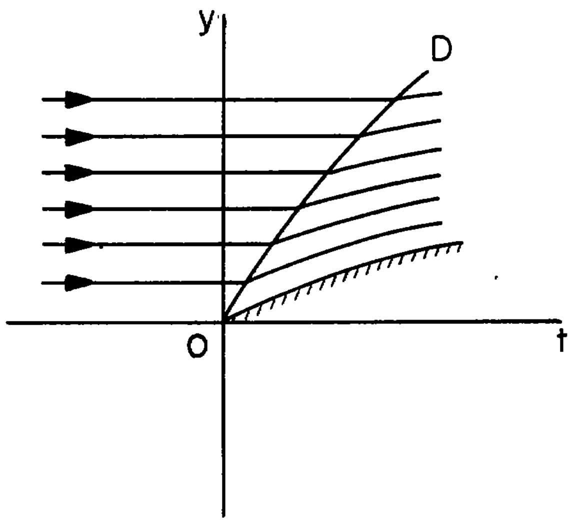

Figure 3.

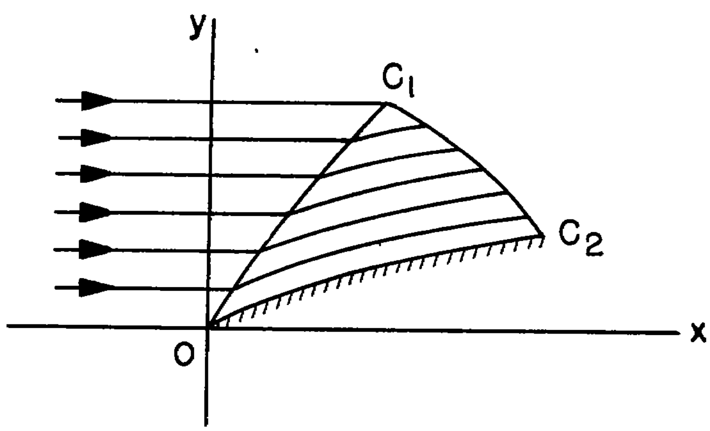

Figure 4.

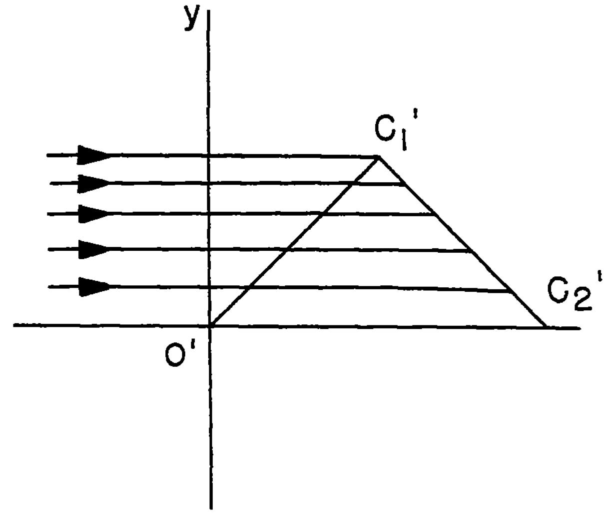

Figure 5.

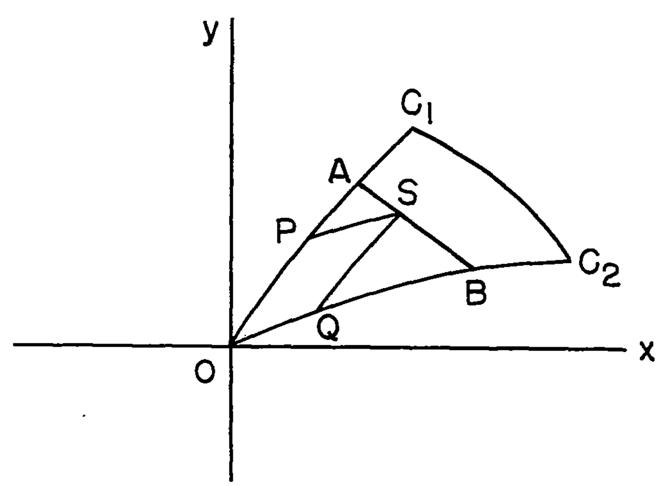

Figure 6.

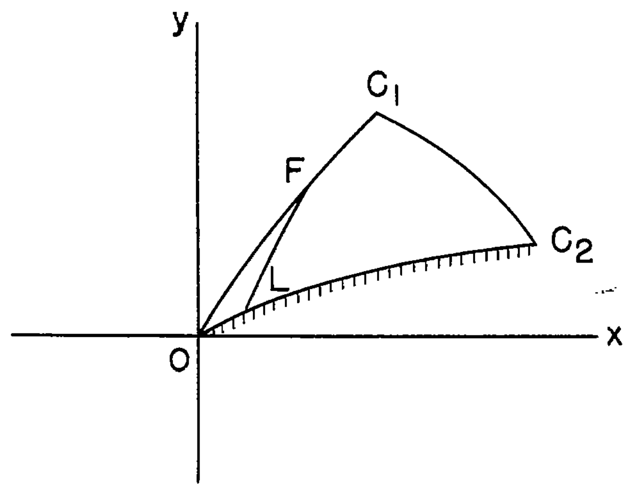

Figure 7.

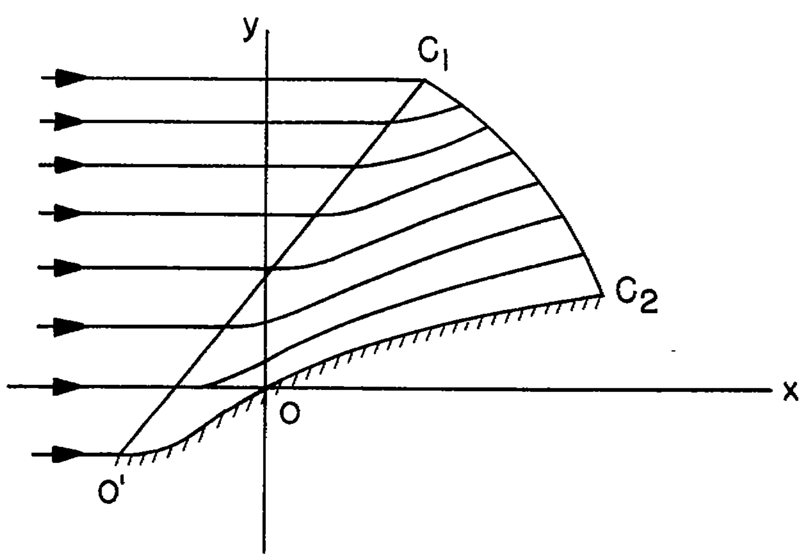

Figure 8.

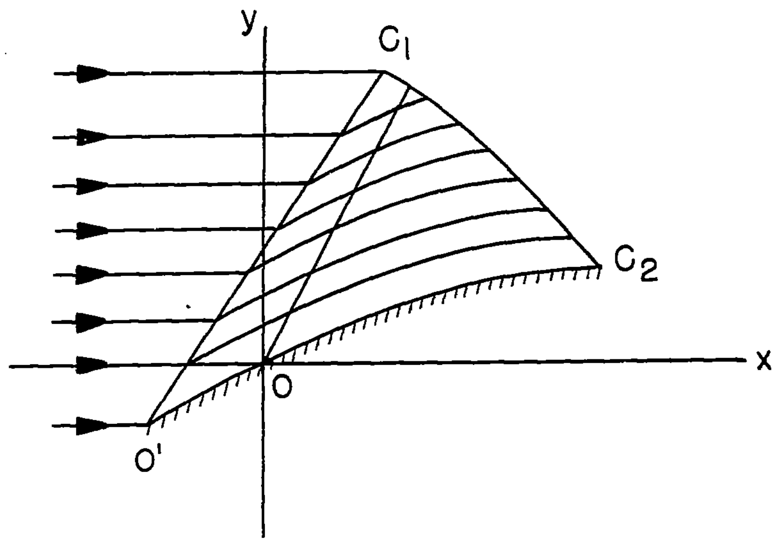

Figure 9.

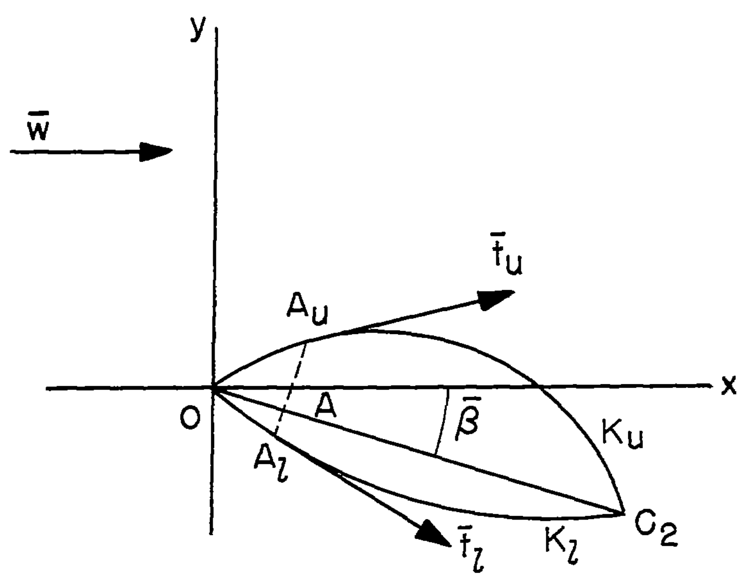

Figure 10.

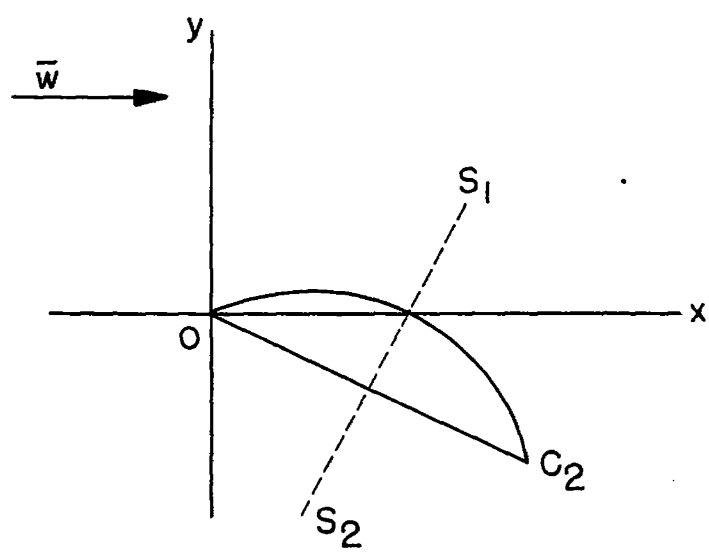

Figure 11.